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COMMENT

On the mixing property of the Fokker–Planck equation

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Abstract. The behaviour of the Fokker-Planck equation in the limit of large times, which has become controversial recently, is discussed in this comment. It is shown, on the basis of a theorem given by Risken, that every solution of the Fokker-Planck equation must tend to the same in this limit, and any information on the initial data will be lost.

The so-called 'stochastic resonance' in a bistationary system has attracted more and more attention in theoretical as well as in experimental researches [1-5]. This phenomenton is characterized by the enhancement of the signal-to-noise ratio caused by injection of noise into a periodically modulated nonlinear system. A theoretical interpretation of this phenomenon leads to the following Fokker-Planck equation

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[f(x)+\varepsilon h(x)\cos(\omega t+\theta)]P(x,t) + D\frac{\partial^2 P(x,t)}{\partial x^2} \tag{1}$$

where

$$f(x) = ax - x^3 \qquad a>0. \tag{2}$$

The solution of this equation in the long-time limit has become controversial recently. Several authors believe that the solution could maintain some initial information [6-8]. In other words, the long-time behaviour is shown to depend on the preparation of the system at the initial time, thus proving the non-mixing nature of the process. Hu *et al* [9, 10], however, proved with the help of a perturbative technique that the solution tends to a unique asymptotic distribution, no matter what initial distribution is chosen. The aim of the present comment is to clarify this questionable point by recalling a theorem given by Risken in 1984 [11]. This theorem concerns the following Fokker-Planck equation

$$\frac{\partial P(\{x\},t)}{\partial t} = -\frac{\partial}{\partial x_i}[A_i(\{x\},t)P(\{x\},t)] + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}[B_{ij}(\{x\},t)P(\{x\},t)] \tag{3}$$

where $\{x\}$ means the set $x_1, x_2, \ldots, x_s$, $A_i(\{x\}, t)$ is an s-dimensional vector, and $B_{ij}(\{x\}, t)$ is a positive definite $s \times s$ tensor. Repeated subscripts imply summation here. This theorem claims that any two solutions $P_1(\{x\}, t)$ and $P_2(\{x\}, t)$ of the equation (3), being normalized positive distribution, will tend to the same after a sufficiently long time, i.e.

$$R = \frac{P_1(\{x\},t)}{P_2(\{x\},t)} \to 1 \qquad \text{as } t \to \infty. \tag{4}$$

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The proof of this theorem can be found in [11, 12] and will not be repeated here. According to this theorem, every solution of the equation (1), as long as the diffusion coefficient $D>0$, must forget all initial information in the limit of long time. This means that the process described by the Fokker–Planck equation is mixing.

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